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CONGRUENCES INVOLVING $\binom{4k}{2k}$ AND $\binom{3k}{k}$

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ABSTRACT. Let p be a prime greater than 3. In the paper we mainly determine $\sum_{k=0}^{[p/4]} \binom{4k}{2k} (-1)^k$, $\sum_{k=0}^{[p/3]} \binom{3k}{k} (-1)^k$ and $\sum_{k=0}^{[p/3]} \binom{3k}{k} (-3)^k$ modulo p , where $[x]$ is the greatest integer not exceeding x .

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1. Introduction.

Recently Z.W. Sun and his coauthors studied congruences involving the binomial coefficient $\binom{3k}{k}$. For example, Zhao, Pan and Sun [15] showed that for any prime $p > 5$,

$$\sum_{k=1}^{p-1} 2^k \binom{3k}{k} \equiv \frac{6}{5} ((-1)^{(p-1)/2} - 1) \pmod{p}.$$

In [13] Z.W. Sun investigated $\sum_{k=0}^{p-1} \binom{3k}{k} m^{-k} \pmod{p}$ for a prime $p > 3$ and $m \not\equiv 0 \pmod{p}$. He gave explicit congruences in the cases $m = 6, 7, 8, 9, 13, -\frac{1}{4}, \frac{4}{27}, \frac{3}{8}$.

Let \mathbb{Z} and \mathbb{N} be the sets of integers and positive integers, respectively. For a prime p let \mathbb{Z}_p denote the set of those rational numbers whose denominator is not divisible by p . Let p be a prime greater than 3 and $m \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod{p}$. In Sections 2 and 3 we study congruences for $\sum_{k=0}^{[p/4]} \binom{4k}{2k} m^k$ and $\sum_{k=0}^{[p/4]} \binom{3k}{k} m^{-k}$ modulo p by using Lucas sequences, binary quadratic forms and the theory of cubic residues, where $[x]$ is the greatest integer not exceeding x . In Section 4 we deduce some congruences involving $\binom{6k}{3k}, \binom{8k}{4k}, \binom{10k}{5k}, \binom{12k}{6k}, \binom{20k}{10k}$ and $\binom{24k}{12k}$ via formulas for the sum $\sum_{k \equiv r \pmod{m}} \binom{n}{k}$ in the cases $m = 3, 4, 5, 6, 10, 12$. For $a, b, c \in \mathbb{Z}$ and a prime p , we say that p is represented by $ax^2 + bxy + cy^2$ or $p = ax^2 + bxy + cy^2$ if there are integers x and y such that $p = ax^2 + bxy + cy^2$. Let $(\frac{a}{n})$ be the Legendre-Jacobi symbol. As examples, we have the following typical results.

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(1.1) Let p be a prime of the form $4k + 3$. Then

$$\sum_{k=0}^{(p-3)/4} (-1)^k \binom{4k}{2k} \equiv \begin{cases} 17^{(p-3)/4} \pmod{p} & \text{if } p \equiv \pm 1, \pm 4 \pmod{17}, \\ -17^{(p-3)/4} \pmod{p} & \text{if } p \equiv \pm 2, \pm 8 \pmod{17}, \\ 4 \cdot 17^{(p-3)/4} \pmod{p} & \text{if } p \equiv \pm 3, \pm 5 \pmod{17}, \\ -4 \cdot 17^{(p-3)/4} \pmod{p} & \text{if } p \equiv \pm 6, \pm 7 \pmod{17}. \end{cases}$$

(1.2) Let p be a prime of the form $4k + 1$ and so $p = c^2 + d^2$ with $c, d \in \mathbb{Z}$ and $2 \mid d$. Let $a \in \mathbb{Z}$ with $p \nmid (16a^2 + 1)$. Then

$$\sum_{k=0}^{(p-1)/4} \binom{4k}{2k} (-a^2)^k \equiv \begin{cases} \left(\frac{c-4ad}{16a^2+1} \right) \pmod{p} & \text{if } \left(\frac{16a^2+1}{p} \right) = 1, \\ 0 \pmod{p} & \text{if } \left(\frac{16a^2+1}{p} \right) = -1. \end{cases}$$

(1.3) Let $p > 3$ be a prime. If $\left(\frac{p}{23} \right) = -1$, then $x \equiv \sum_{k=0}^{[p/3]} \binom{3k}{k} \pmod{p}$ is the unique solution of the congruence $23x^3 + 3x + 1 \equiv 0 \pmod{p}$. If $\left(\frac{p}{23} \right) = 1$, then

$$\sum_{k=0}^{[p/3]} \binom{3k}{k} \equiv \begin{cases} 1 \pmod{p} & \text{if } p = x^2 + xy + 52y^2, \ 8x^2 + 7xy + 8y^2, \\ (39x - 10y)/(23y) \pmod{p} & \text{if } p = 13x^2 + xy + 4y^2 \neq 13, \\ -(87x + 19y)/(23y) \pmod{p} & \text{if } p = 29x^2 + 5xy + 2y^2 \neq 29. \end{cases}$$

(1.4) Let p be a prime with $p \equiv 13 \pmod{24}$. Then $\sum_{k=0}^{(p-1)/12} \binom{12k}{6k} \frac{1}{(-4096)^k} \equiv 0 \pmod{p}$.

2. Congruences involving $\binom{4k}{2k}$.

For any numbers P and Q , let $\{U_n(P, Q)\}$ and $\{V_n(P, Q)\}$ be the Lucas sequences given by

$$U_0 = 0, \ U_1 = 1, \ U_{n+1} = PU_n - QU_{n-1} \ (n \geq 1)$$

and

$$V_0 = 2, \ V_1 = P, \ V_{n+1} = PV_n - QV_{n-1} \ (n \geq 1).$$

It is well known that (see [14])

$$U_n(P, Q) = \begin{cases} \frac{1}{\sqrt{P^2-4Q}} \left\{ \left(\frac{P+\sqrt{P^2-4Q}}{2} \right)^n - \left(\frac{P-\sqrt{P^2-4Q}}{2} \right)^n \right\} & \text{if } P^2 - 4Q \neq 0, \\ n \left(\frac{P}{2} \right)^{n-1} & \text{if } P^2 - 4Q = 0 \end{cases}$$

and

$$V_n(P, Q) = \left(\frac{P + \sqrt{P^2 - 4Q}}{2} \right)^n + \left(\frac{P - \sqrt{P^2 - 4Q}}{2} \right)^n.$$

In particular, we have

$$(2.1) \quad U_n(2, 1) = n \quad \text{and} \quad U_n(a + b, ab) = \frac{a^n - b^n}{a - b} \quad \text{for } a \neq b.$$

As usual, the sequences $F_n = U_n(1, -1)$ and $L_n = V_n(1, -1)$ are called the Fibonacci sequence and the Lucas sequence, respectively. It is easily seen that (see [3, Lemma 1.7])

$$(2.2) \quad 2U_{n+1}(P, Q) = PU_n(P, Q) + V_n(P, Q), \quad 2QU_{n-1}(P, Q) = PU_n(P, Q) - V_n(P, Q).$$

Lemma 2.1([14, (4.2.39)]). For $n \in \mathbb{N}$ we have

$$U_{2n+1}(P, Q) = \sum_{k=0}^n \binom{n+k}{n-k} (-Q)^{n-k} P^{2k}.$$

Lemma 2.2. Let p be an odd prime and $k \in \{1, 2, \dots, \lfloor \frac{p}{4} \rfloor\}$. Then

$$\binom{\lfloor \frac{p}{4} \rfloor + k}{\lfloor \frac{p}{4} \rfloor - k} \equiv \binom{4k}{2k} \frac{1}{(-64)^k} \pmod{p}.$$

Proof. Suppose $r = 1$ or 3 according as $4 \mid p-1$ or $4 \mid p-3$. Then clearly

$$\begin{aligned} \binom{\frac{p-r}{4} + k}{2k} &= \frac{(\frac{p-r}{4} + k)(\frac{p-r}{4} + k - 1) \cdots (\frac{p-r}{4} - k + 1)}{(2k)!} \\ &= \frac{(p + 4k - r)(p + 4k - r - 4) \cdots (p - (4k + r - 4))}{4^{2k} \cdot (2k)!} \\ &\equiv (-1)^k \frac{(4k - r)(4k - r - 4) \cdots (4 - r) \cdot r(r + 4) \cdots (4k + r - 4)}{4^{2k} \cdot (2k)!} \\ &= \frac{(-1)^k \cdot (4k)!}{2^{2k} \cdot (2k)! \cdot 4^{2k} \cdot (2k)!} = \frac{\binom{4k}{2k}}{(-64)^k} \pmod{p}. \end{aligned}$$

So the result follows.

Lemma 2.3. Let p be an odd prime and $k \in \{1, 2, \dots, \frac{p-1}{2}\}$. Then

$$\binom{(p-1)/2}{k} \equiv \frac{1}{(-4)^k} \binom{2k}{k} \pmod{p}.$$

Proof. It is clear that

$$\begin{aligned} \binom{\frac{p-1}{2}}{k} &= \frac{\frac{p-1}{2}(\frac{p-1}{2} - 1) \cdots (\frac{p-1}{2} - k + 1)}{k!} = \frac{(p-1)(p-3) \cdots (p - (2k-1))}{2^k \cdot k!} \\ &\equiv \frac{(-1)(-3) \cdots (-(2k-1))}{2^k \cdot k!} = \frac{(-1)^k \cdot (2k)!}{(2^k \cdot k!)^2} \pmod{p}. \end{aligned}$$

This yields the result.

Theorem 2.1. Let p be an odd prime and $P, Q \in \mathbb{Z}_p$ with $p \nmid PQ$. Then

$$\sum_{k=0}^{\lfloor p/4 \rfloor} \binom{4k}{2k} \left(\frac{P^2}{64Q} \right)^k \equiv (-Q)^{-\lfloor p/4 \rfloor} U_{\frac{p+(-1)}{2}}(P, Q) \pmod{p}$$

and

$$\sum_{k=0}^{\lfloor p/4 \rfloor} \binom{4k}{2k} \left(\frac{Q}{4P^2} \right)^k \equiv \left(\frac{P}{p} \right) U_{\frac{p+1}{2}}(P, Q) \pmod{p}.$$

Proof. Using Lemmas 2.1 and 2.2 we see that

$$\begin{aligned} U_{2[\frac{p}{4}]+1}(P, Q) &= \sum_{k=0}^{[p/4]} \binom{[\frac{p}{4}] + k}{[\frac{p}{4}] - k} (-Q)^{[\frac{p}{4}] - k} P^{2k} \\ &\equiv (-Q)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k}}{(-64)^k} \left(\frac{P^2}{-Q}\right)^k \pmod{p}. \end{aligned}$$

Note that $2[\frac{p}{4}] + 1 = (p + (\frac{-1}{p}))/2$. We deuce the first result.

Using Lemma 2.3 we see that

$$\begin{aligned} V_{\frac{p-1}{2}}(P, (P^2 - 4Q)/4) &= \left(\frac{P + 2\sqrt{Q}}{2}\right)^{\frac{p-1}{2}} + \left(\frac{P - 2\sqrt{Q}}{2}\right)^{\frac{p-1}{2}} \\ &= \frac{1}{2^{\frac{p-1}{2}}} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k} P^{\frac{p-1}{2} - k} ((2\sqrt{Q})^k + (-2\sqrt{Q})^k) \\ &= \frac{2}{2^{\frac{p-1}{2}}} \sum_{k=0}^{[p/4]} \binom{\frac{p-1}{2}}{2k} P^{\frac{p-1}{2} - 2k} (2\sqrt{Q})^{2k} \\ &\equiv \frac{2}{2^{\frac{p-1}{2}}} \sum_{k=0}^{[p/4]} \binom{4k}{2k} (-4)^{-2k} P^{\frac{p-1}{2} - 2k} (4Q)^k \\ &\equiv 2 \left(\frac{2P}{p}\right) \sum_{k=0}^{[p/4]} \binom{4k}{2k} \left(\frac{Q}{4P^2}\right)^k \pmod{p}. \end{aligned}$$

By appealing to [7, Lemma 3.1] we have

$$U_{\frac{p+1}{2}}(P, Q) \equiv \frac{1}{2} \left(\frac{2}{p}\right) V_{\frac{p-1}{2}}(P, (P^2 - 4Q)/4) \equiv \left(\frac{P}{p}\right) \sum_{k=0}^{[p/4]} \binom{4k}{2k} \left(\frac{Q}{4P^2}\right)^k \pmod{p}.$$

This completes the proof.

Lemma 2.4. For $n \in \mathbb{N}$ we have $U_n(1, 1) = (-1)^{n-1} \left(\frac{n}{3}\right)$.

Proof. Set $\omega = (-1 + \sqrt{-3})/2$. Then

$$\begin{aligned} U_n(1, 1) &= \frac{1}{\sqrt{-3}} \left\{ \left(\frac{1 + \sqrt{-3}}{2}\right)^n - \left(\frac{1 - \sqrt{-3}}{2}\right)^n \right\} \\ &= \frac{(-1)^n}{\sqrt{-3}} \left\{ \left(\frac{-1 - \sqrt{-3}}{2}\right)^n - \left(\frac{-1 + \sqrt{-3}}{2}\right)^n \right\} \\ &= \frac{(-1)^n}{\omega(1 - \omega)} (\omega^{2n} - \omega^n) = \begin{cases} 0 & \text{if } 3 \mid n \\ (-1)^{n-1} & \text{if } 3 \mid n - 1 \\ (-1)^n & \text{if } 3 \mid n - 2. \end{cases} \end{aligned}$$

This yields the result.

Corollary 2.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k} \frac{1}{16^k} \equiv \frac{1}{2} \left(\frac{2}{p} \right) \pmod{p},$$

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k} \frac{1}{4^k} \equiv \begin{cases} (-1)^{\frac{p-1}{2}} \pmod{p} & \text{if } p \equiv 1 \pmod{3}, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3} \end{cases}$$

and

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k} \frac{1}{64^k} \equiv \begin{cases} 0 \pmod{p} & \text{if } p \equiv 5, 7, 17, 19 \pmod{24}, \\ 1 \pmod{p} & \text{if } p \equiv 1, 23 \pmod{24}, \\ -1 \pmod{p} & \text{if } p \equiv 11, 13 \pmod{24}. \end{cases}$$

Proof. Taking $(P, Q) = (2, 1), (1, 1)$ in Theorem 2.1 and then applying (2.1) and Lemma 2.4 we deduce the result.

Theorem 2.2. *Let p be an odd prime and $x \in \mathbb{Z}_p$ with $x \not\equiv 0, 1 \pmod{p}$.*

(i) *If $p \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k} \left(\frac{x}{16} \right)^k \equiv x^{\frac{p-1}{4}} \sum_{k=0}^{[p/4]} \binom{4k}{2k} \frac{1}{(16x)^k} \pmod{p}.$$

(ii) *If $p \equiv 3 \pmod{4}$, then*

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k} \frac{1}{(16x)^k} \equiv \left(1 - \frac{1}{x} \right)^{\frac{p-3}{4}} \sum_{k=0}^{[p/4]} \binom{4k}{2k} \frac{1}{(16(1-x))^k} \pmod{p}.$$

Proof. If $p \equiv 1 \pmod{4}$, by Theorem 2.1 we have

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k} \left(\frac{x}{16} \right)^k \equiv \left(\frac{2}{p} \right) U_{\frac{p+1}{2}}(2, x) \equiv (-1)^{\frac{p-1}{4}} \cdot (-x)^{\frac{p-1}{4}} \sum_{k=0}^{[p/4]} \binom{4k}{2k} \frac{1}{(16x)^k} \pmod{p}.$$

If $p \equiv 3 \pmod{4}$, from [7, Lemma 3.1] we know that $U_{\frac{p-1}{2}}(2, x) \equiv -\left(\frac{2}{p}\right)U_{\frac{p-1}{2}}(2, 1-x) \pmod{p}$. Now applying Theorem 2.1 and the fact $\left(\frac{2}{p}\right) = -(-1)^{(p-3)/4}$ we deduce

$$\begin{aligned} \sum_{k=0}^{[p/4]} \binom{4k}{2k} \frac{1}{(16x)^k} &\equiv (-x)^{-[p/4]} U_{\frac{p-1}{2}}(2, x) \equiv x^{-[p/4]} U_{\frac{p-1}{2}}(2, 1-x) \\ &\equiv (x-1)^{[\frac{p}{4}]} x^{-[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \binom{4k}{2k} \frac{1}{(16(1-x))^k} \pmod{p}. \end{aligned}$$

So the theorem is proved.

Corollary 2.2. *Let p be a prime of the form $8k + 7$. Then*

$$\sum_{k=0}^{(p-3)/4} \binom{4k}{2k} 8^{-k} \equiv 0 \pmod{p}.$$

Proof. Taking $x = \frac{1}{2}$ in Theorem 2.2(ii) we deduce the result.

Theorem 2.3. *Let p be an odd prime and $P, Q \in \mathbb{Z}_p$ with $p \nmid PQ(P^2 - 4Q)$ and $(\frac{Q}{p}) = 1$.*

- (i) *If $(\frac{4Q-P^2}{p}) = -1$, then $\sum_{k=0}^{[p/4]} \binom{4k}{2k} (\frac{P^2}{64Q})^k \equiv 0 \pmod{p}$.*
- (ii) *If $(\frac{P^2-4Q}{p}) = -1$, then $\sum_{k=0}^{[p/4]} \binom{4k}{2k} (\frac{Q}{4P^2})^k \equiv 0 \pmod{p}$.*

Proof. Since $(\frac{Q}{p}) = 1$, it is well known that (see [3]) $U_{(p-(\frac{P^2-4Q}{p}))/2}(P, Q) \equiv 0 \pmod{p}$. This together with Theorem 2.1 yields the result.

Theorem 2.4. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k}}{(-16)^k} \equiv \begin{cases} (-1)^{\frac{p-1}{8}} 2^{\frac{p-1}{4}} \pmod{p} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{\frac{p-3}{8}} 2^{\frac{p-3}{4}} \pmod{p} & \text{if } p \equiv 3 \pmod{8}, \\ 0 \pmod{p} & \text{if } p \equiv 5 \pmod{8}, \\ (-1)^{\frac{p+1}{8}} 2^{\frac{p-3}{4}} \pmod{p} & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Proof. Taking $P = 2$ and $Q = -1$ in Theorem 2.1 we obtain

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k} (-16)^{-k} \equiv U_{(p+(\frac{-1}{p}))/2}(2, -1) \pmod{p}.$$

Now applying [4, Theorem 2.3] we deduce the result.

Theorem 2.5. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k}}{(-64)^k} \equiv \begin{cases} (-1)^{[\frac{p+5}{10}]} 5^{[\frac{p}{4}]} \pmod{p} & \text{if } p \equiv 1, 3, 7, 9 \pmod{20}, \\ 2(-1)^{[\frac{p+5}{10}]} 5^{\frac{p-3}{4}} \pmod{p} & \text{if } p \equiv 11, 19 \pmod{20}, \\ 0 \pmod{p} & \text{if } p \equiv 13, 17 \pmod{20} \end{cases}$$

and

$$\sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k}}{(-4)^k} \equiv \begin{cases} (-1)^{[\frac{p+5}{10}]} 5^{[\frac{p}{4}]} \pmod{p} & \text{if } p \equiv 1, 9, 11, 19 \pmod{20}, \\ -2(-1)^{[\frac{p+5}{10}]} 5^{\frac{p-3}{4}} \pmod{p} & \text{if } p \equiv 3, 7 \pmod{20}, \\ 0 \pmod{p} & \text{if } p \equiv 13, 17 \pmod{20}. \end{cases}$$

Proof. Taking $P = 1$ and $Q = -1$ in Theorem 2.1 we obtain

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k} (-64)^{-k} \equiv F_{\frac{p+(\frac{-1}{p})}{2}} \pmod{p} \quad \text{and} \quad \sum_{k=0}^{[p/4]} \binom{4k}{2k} (-4)^{-k} \equiv F_{\frac{p+1}{2}} \pmod{p}.$$

Now applying [11, Corollaries 1 and 2] we deduce the result.

Theorem 2.6. *Let p be an odd prime with $p \neq 17$.*

(i) *If $p \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{[p/4]} (-1)^k \binom{4k}{2k} \equiv \begin{cases} 0 \pmod{p} & \text{if } p \equiv \pm 3, \pm 5, \pm 6, \pm 7 \pmod{17}, \\ 17^{(p-1)/4} \pmod{p} & \text{if } p \equiv \pm 1, \pm 4 \pmod{17}, \\ -17^{(p-1)/4} \pmod{p} & \text{if } p \equiv \pm 2, \pm 8 \pmod{17}. \end{cases}$$

(ii) *If $p \equiv 3 \pmod{4}$, then*

$$\sum_{k=0}^{[p/4]} (-1)^k \binom{4k}{2k} \equiv \begin{cases} 17^{(p-3)/4} \pmod{p} & \text{if } p \equiv \pm 1, \pm 4 \pmod{17}, \\ -17^{(p-3)/4} \pmod{p} & \text{if } p \equiv \pm 2, \pm 8 \pmod{17}, \\ 4 \cdot 17^{(p-3)/4} \pmod{p} & \text{if } p \equiv \pm 3, \pm 5 \pmod{17}, \\ -4 \cdot 17^{(p-3)/4} \pmod{p} & \text{if } p \equiv \pm 6, \pm 7 \pmod{17}. \end{cases}$$

Proof. Taking $P = 8$ and $Q = -1$ in Theorem 2.1 we see that

$$(2.3) \quad \sum_{k=0}^{[p/4]} (-1)^k \binom{4k}{2k} \equiv U_{\frac{p+(-1)}{2}}(8, -1) \pmod{p}.$$

By (2.2) we have $U_{\frac{p+1}{2}}(8, -1) = 4U_{\frac{p-1}{2}}(8, -1) + \frac{1}{2}V_{\frac{p-1}{2}}(8, -1)$. From the above and [10, Corollary 4.5] we deduce the result.

Theorem 2.7. *Let p be an odd prime with $p \neq 3, 13$. Then*

$$\left(\frac{3}{p}\right) \sum_{k=0}^{[p/4]} \binom{4k}{2k} \frac{1}{(-36)^k} \equiv \begin{cases} 13^{\frac{p-1}{4}} \pmod{p} & \text{if } p \equiv 1, 9, -23 \pmod{52}, \\ -13^{\frac{p-1}{4}} \pmod{p} & \text{if } p \equiv -3, 17, 25 \pmod{52}, \\ 3 \cdot 13^{\frac{p-3}{4}} \pmod{p} & \text{if } p \equiv -1, -9, 23 \pmod{52}, \\ -3 \cdot 13^{\frac{p-3}{4}} \pmod{p} & \text{if } p \equiv 3, -17, -25 \pmod{52}, \\ 0 \pmod{p} & \text{if } p \equiv 5, -7, -11, -15, -19, 21 \pmod{52}, \\ 2 \cdot 13^{\frac{p-3}{4}} \pmod{p} & \text{if } p \equiv -5, 7, 11 \pmod{52}, \\ -2 \cdot 13^{\frac{p-3}{4}} \pmod{p} & \text{if } p \equiv 15, 19, -21 \pmod{52}. \end{cases}$$

Proof. Taking $P = 3$ and $Q = -1$ in Theorem 2.1 we see that

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k} \frac{1}{(-36)^k} \equiv \left(\frac{3}{p}\right) U_{\frac{p+1}{2}}(3, -1) \pmod{p}.$$

Now applying [7, Corollary 4.1] we deduce the result.

Theorem 2.8. *Let p be a prime such that $p \equiv 1, 9, 11, 13, 19, 37 \pmod{40}$. Then*

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k} \frac{1}{(-144)^k} \equiv \begin{cases} (-1)^{[\frac{p}{3}] + \frac{y}{2}} \pmod{p} & \text{if } p = x^2 + 10y^2 \equiv 1, 9 \pmod{40}, \\ (-1)^{[\frac{p}{3}] \frac{y}{x}} \pmod{p} & \text{if } p = x^2 + 10y^2 \equiv 11, 19 \pmod{40} \text{ and } 4 \mid x - y, \\ (-1)^{[\frac{p}{3}] + \frac{y}{2}} \pmod{p} & \text{if } p = 5x^2 + 2y^2 \equiv 13, 37 \pmod{40}. \end{cases}$$

Proof. Taking $P = 6$ and $Q = -1$ in Theorem 2.1 we see that

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k} \frac{1}{(-144)^k} \equiv \left(\frac{6}{p}\right) U_{\frac{p+1}{2}}(6, -1) \pmod{p}.$$

Now applying [10, Theorem 5.4] we deduce the result.

Lemma 2.5 ([5, Lemma 3.4]). *Let p be an odd prime and $P, Q \in \mathbb{Z}_p$ with $p \nmid Q(P^2 - 4Q)$. If $(\frac{Q}{p}) = 1$ and $c^2 \equiv Q \pmod{p}$ for $c \in \mathbb{Z}_p$, then*

$$U_{\frac{p+1}{2}}(P, Q) \equiv \begin{cases} \left(\frac{P-2c}{p}\right) \pmod{p} & \text{if } \left(\frac{P^2-4Q}{p}\right) = 1, \\ 0 \pmod{p} & \text{if } \left(\frac{P^2-4Q}{p}\right) = -1 \end{cases}$$

and

$$U_{\frac{p-1}{2}}(P, Q) \equiv \begin{cases} 0 \pmod{p} & \text{if } \left(\frac{P^2-4Q}{p}\right) = 1, \\ \frac{1}{c} \left(\frac{P-2c}{p}\right) \pmod{p} & \text{if } \left(\frac{P^2-4Q}{p}\right) = -1. \end{cases}$$

Theorem 2.9. *Let p be an odd prime and $a \in \mathbb{Z}_p$ with $p \nmid (16a^2 - 1)$. Then*

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k} a^{2k} \equiv \begin{cases} 0 \pmod{p} & \text{if } \left(\frac{1-16a^2}{p}\right) = -1, \\ \left(\frac{1-4a}{p}\right) \pmod{p} & \text{if } \left(\frac{1-16a^2}{p}\right) = 1. \end{cases}$$

Proof. Putting $P = 8a$ and $Q = 1$ in Theorem 2.1 we deduce that

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k} a^{2k} \equiv (-1)^{[p/4]} U_{\frac{p+(-1)}{2}}(8a, 1) \pmod{p}.$$

If $p \equiv 1 \pmod{4}$, by Lemma 2.5 we have

$$(-1)^{\frac{p-1}{4}} U_{\frac{p+1}{2}}(8a, 1) \equiv \begin{cases} (-1)^{\frac{p-1}{4}} \left(\frac{8a-2}{p}\right) = \left(\frac{1-4a}{p}\right) \pmod{p} & \text{if } \left(\frac{16a^2-1}{p}\right) = 1, \\ 0 \pmod{p} & \text{if } \left(\frac{16a^2-1}{p}\right) = -1. \end{cases}$$

If $p \equiv 3 \pmod{4}$, by Lemma 2.5 we have

$$(-1)^{\frac{p-3}{4}} U_{\frac{p-1}{2}}(8a, 1) \equiv \begin{cases} (-1)^{\frac{p-3}{4}} \left(\frac{8a-2}{p}\right) = \left(\frac{1-4a}{p}\right) \pmod{p} & \text{if } \left(\frac{16a^2-1}{p}\right) = -1, \\ 0 \pmod{p} & \text{if } \left(\frac{16a^2-1}{p}\right) = 1. \end{cases}$$

Now combining all the above we obtain the result.

Corollary 2.2. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{\lfloor p/4 \rfloor} \binom{4k}{2k} \equiv \begin{cases} 0 \pmod{p} & \text{if } p \equiv 7, 11, 13, 14 \pmod{15}, \\ 1 \pmod{p} & \text{if } p \equiv 1, 4 \pmod{15}, \\ -1 \pmod{p} & \text{if } p \equiv 2, 8 \pmod{15}. \end{cases}$$

Proof. Since $(\frac{-15}{p}) = (\frac{p}{15}) = (\frac{p}{3})(\frac{p}{5})$, putting $a = 1$ in Theorem 2.9 we deduce the result.

Corollary 2.3. *Let $p > 7$ be a prime. Then*

$$\sum_{k=0}^{\lfloor p/4 \rfloor} \binom{4k}{2k} 2^{2k} \equiv \begin{cases} 1 \pmod{p} & \text{if } p \equiv 1, 2, 4 \pmod{7}, \\ 0 \pmod{p} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

Proof. Since $(\frac{-63}{p}) = (\frac{-7}{p}) = (\frac{p}{7})$, putting $a = 2$ in Theorem 2.9 we deduce the result.

Corollary 2.4. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{\lfloor p/4 \rfloor} \binom{4k}{2k} \frac{1}{2^{2k}} \equiv \begin{cases} (-1)^{(p-1)/2} \pmod{p} & \text{if } p \equiv 1 \pmod{3}, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Proof. Taking $a = \frac{1}{2}$ in Theorem 2.9 we deduce the result.

Theorem 2.10. *Let p be a prime of the form $4k + 1$ and $p = c^2 + d^2$ with $c, d \in \mathbb{Z}$ and $2 \mid d$. Let $b, m \in \mathbb{Z}$ with $\gcd(b, m) = 1$ and $p \nmid m(b^2 + 4m^2)$. Then*

$$\begin{aligned} \left(\frac{m}{p}\right) \sum_{k=0}^{(p-1)/4} \binom{4k}{2k} \left(-\frac{b^2}{64m^2}\right)^k &\equiv \left(\frac{b}{p}\right) \sum_{k=0}^{(p-1)/4} \binom{4k}{2k} \left(-\frac{m^2}{4b^2}\right)^k \\ &\equiv \begin{cases} \left(\frac{bc+2md}{b^2+4m^2}\right) \pmod{p} & \text{if } 2 \nmid b \text{ and } \left(\frac{b^2+4m^2}{p}\right) = 1, \\ (-1)^{\frac{(\frac{b}{2}c+md)^2-1}{8} + \frac{d}{2}} \left(\frac{\frac{b}{2}c+md}{((\frac{b}{2})^2+m^2)/2}\right) \pmod{p} & \text{if } 2 \parallel b \text{ and } \left(\frac{b^2+4m^2}{p}\right) = 1, \\ \left(\frac{mc-\frac{b}{2}d}{\frac{b^2}{4}+m^2}\right) \pmod{p} & \text{if } 4 \mid b \text{ and } \left(\frac{b^2+4m^2}{p}\right) = 1, \\ 0 \pmod{p} & \text{if } \left(\frac{b^2+4m^2}{p}\right) = -1. \end{cases} \end{aligned}$$

In particular, taking $b = 8a$ and $m = 1$ we have

$$\begin{aligned} \sum_{k=0}^{(p-1)/4} \binom{4k}{2k} (-a^2)^k &\equiv \left(\frac{2a}{p}\right) \sum_{k=0}^{(p-1)/4} \binom{4k}{2k} \frac{1}{(-256a^2)^k} \\ &\equiv \begin{cases} \left(\frac{c-4ad}{16a^2+1}\right) \pmod{p} & \text{if } \left(\frac{16a^2+1}{p}\right) = 1, \\ 0 \pmod{p} & \text{if } \left(\frac{16a^2+1}{p}\right) = -1. \end{cases} \end{aligned}$$

Proof. Putting $P = b$ and $Q = -m^2$ in Theorem 2.1 we see that

$$U_{\frac{p+1}{2}}(b, -m^2) \equiv m^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{4}} \binom{4k}{2k} \left(-\frac{b^2}{64m^2}\right)^k \equiv \left(\frac{b}{p}\right) \sum_{k=0}^{\frac{p-1}{4}} \binom{4k}{2k} \left(-\frac{m^2}{4b^2}\right)^k \pmod{p}.$$

Now applying [10, Theorem 3.2] we deduce the result.

Corollary 2.5. *Let p be a prime with $p \equiv 1 \pmod{4}$ and $p \equiv \pm 1, \pm 2, \pm 4, \pm 8 \pmod{17}$. Let $p = c^2 + d^2$ with $c, d \in \mathbb{Z}$ and $2 \mid d$. Then*

$$\sum_{k=0}^{(p-1)/4} (-1)^k \binom{4k}{2k} \equiv \left(\frac{c-4d}{17} \right) \pmod{p}.$$

Proof. Taking $a = 1$ in Theorem 2.10 we obtain the result.

Theorem 2.11. *Let p be an odd prime, and $b, m \in \mathbb{Z}$ with $p \nmid bm(b^2 + 4m^2)$. Then there is a unique $\delta_p \in \{1, -1\}$ such that*

$$\begin{aligned} & \left(\frac{b}{p} \right) \sum_{k=0}^{[p/4]} \binom{4k}{2k} \left(-\frac{m^2}{4b^2} \right)^k \\ & \equiv \begin{cases} \delta_p (b^2 + 4m^2)^{\frac{p-1}{4}} \pmod{p} & \text{if } 4 \mid p-1 \text{ and } \left(\frac{b^2+4m^2}{p} \right) = 1, \\ b\delta_p (b^2 + 4m^2)^{\frac{p-3}{4}} \pmod{p} & \text{if } 4 \mid p-3 \text{ and } \left(\frac{b^2+4m^2}{p} \right) = 1, \\ 0 \pmod{p} & \text{if } 4 \mid p-1 \text{ and } \left(\frac{b^2+4m^2}{p} \right) = -1, \\ 2m\delta_p (b^2 + 4m^2)^{\frac{p-3}{4}} \pmod{p} & \text{if } 4 \mid p-3 \text{ and } \left(\frac{b^2+4m^2}{p} \right) = -1 \end{cases} \end{aligned}$$

and

$$\begin{aligned} & \left(\frac{m}{p} \right) \sum_{k=0}^{[p/4]} \binom{4k}{2k} \left(-\frac{b^2}{64m^2} \right)^k \\ & \equiv \begin{cases} \delta_p (b^2 + 4m^2)^{\frac{p-1}{4}} \pmod{p} & \text{if } 4 \mid p-1 \text{ and } \left(\frac{b^2+4m^2}{p} \right) = 1, \\ 2m\delta_p (b^2 + 4m^2)^{\frac{p-3}{4}} \pmod{p} & \text{if } 4 \mid p-3 \text{ and } \left(\frac{b^2+4m^2}{p} \right) = 1, \\ 0 \pmod{p} & \text{if } 4 \mid p-1 \text{ and } \left(\frac{b^2+4m^2}{p} \right) = -1, \\ -b\delta_p (b^2 + 4m^2)^{\frac{p-3}{4}} \pmod{p} & \text{if } 4 \mid p-3 \text{ and } \left(\frac{b^2+4m^2}{p} \right) = -1. \end{cases} \end{aligned}$$

Moreover, if p' is also an odd prime satisfying $p' \nmid bm(b^2 + 4m^2)$ and $p' \equiv \pm p \pmod{(3 - (-1)^b)(b^2 + 4m^2)}$, then $\delta_p = \delta_{p'}$. Indeed,

$$\delta_p = \begin{cases} \left(\frac{b+2mi}{p} \right)_4 & \text{if } \left(\frac{b^2+4m^2}{p} \right) = 1, \\ \left(\frac{b+2mi}{p} \right)_4 i & \text{if } \left(\frac{b^2+4m^2}{p} \right) = -1, \end{cases}$$

where $\left(\frac{b+ci}{p} \right)_4$ is the quartic Jacobi symbol.

Proof. Putting $P = b$ and $Q = -m^2$ in Theorem 2.1 and then applying [7, Theorem 4.1] we deduce the result.

Theorem 2.12. *Let p be a prime of the form $4k+1$ and $a \in \mathbb{Z}$ with $p \nmid (1+16a^2)(1-16a^2)$. Let $p = c^2 + d^2$ with $c, d \in \mathbb{Z}$ and $2 \mid d$. Then*

$$2 \sum_{k=0}^{[p/8]} \binom{8k}{4k} a^{4k} \equiv \begin{cases} \left(\frac{1-4a}{p} \right) + \left(\frac{c-4ad}{16a^2+1} \right) \pmod{p} & \text{if } \left(\frac{1-16a^2}{p} \right) = \left(\frac{1+16a^2}{p} \right) = 1, \\ \left(\frac{1-4a}{p} \right) \pmod{p} & \text{if } \left(\frac{1-16a^2}{p} \right) = -\left(\frac{1+16a^2}{p} \right) = 1, \\ \left(\frac{c-4ad}{16a^2+1} \right) \pmod{p} & \text{if } \left(\frac{1-16a^2}{p} \right) = -\left(\frac{1+16a^2}{p} \right) = -1, \\ 0 \pmod{p} & \text{if } \left(\frac{1-16a^2}{p} \right) = \left(\frac{1+16a^2}{p} \right) = -1. \end{cases}$$

Proof. Since

$$2 \sum_{k=0}^{[p/8]} \binom{8k}{4k} a^{2k} = \sum_{k=0}^{(p-1)/4} \binom{4k}{2k} a^{2k} + \sum_{k=0}^{(p-1)/4} \binom{4k}{2k} (-1)^k a^{2k},$$

from Theorems 2.9 and 2.10 we deduce the result.

Corollary 2.7. *Let p be a prime of the form $4k+1$ with $p \neq 5, 17$. Let $p = c^2 + d^2$ with $c, d \in \mathbb{Z}$ and $2 \mid d$. Then*

$$2 \sum_{k=0}^{[p/8]} \binom{8k}{4k} \equiv \begin{cases} \left(\frac{p}{3}\right) + \left(\frac{c-4d}{17}\right) \pmod{p} & \text{if } \left(\frac{p}{15}\right) = \left(\frac{p}{17}\right) = 1, \\ \left(\frac{p}{3}\right) \pmod{p} & \text{if } \left(\frac{p}{15}\right) = -\left(\frac{p}{17}\right) = 1, \\ \left(\frac{c-4d}{17}\right) \pmod{p} & \text{if } \left(\frac{p}{15}\right) = -\left(\frac{p}{17}\right) = -1, \\ 0 \pmod{p} & \text{if } \left(\frac{p}{15}\right) = \left(\frac{p}{17}\right) = -1. \end{cases}$$

Proof. Taking $a = 1$ in Theorem 2.12 we obtain the result.

3. Congruences involving $\binom{3k}{k}$.

Lemma 3.1. *Let $p > 3$ be a prime and $k \in \{1, 2, \dots, [\frac{p}{3}]\}$. Then*

$$\binom{[\frac{p}{3}] + k}{[\frac{p}{3}] - k} \equiv \binom{3k}{k} \frac{1}{(-27)^k} \pmod{p}.$$

Proof. Suppose $r = 1$ or 2 according as $3 \mid p-1$ or $3 \mid p-2$. Then clearly

$$\begin{aligned} \binom{\frac{p-r}{3} + k}{2k} &= \frac{(\frac{p-r}{3} + k)(\frac{p-r}{3} + k - 1) \cdots (\frac{p-r}{3} - k + 1)}{(2k)!} \\ &= \frac{(p + 3k - r)(p + 3k - r - 3) \cdots (p - (3k + r - 3))}{3^{2k} \cdot (2k)!} \\ &\equiv (-1)^k \frac{(3k - r)(3k - r - 3) \cdots (3 - r) \cdot r(r + 3) \cdots (3k + r - 3)}{3^{2k} \cdot (2k)!} \\ &= \frac{(-1)^k \cdot (3k)!}{3 \cdot 6 \cdots 3k \cdot 3^{2k} \cdot (2k)!} = \frac{(-1)^k \cdot (3k)!}{3^k \cdot k! \cdot 3^{2k} \cdot (2k)!} \pmod{p}. \end{aligned}$$

So the result follows.

Theorem 3.1. *Let $p > 3$ be a prime and $a, b \in \mathbb{Z}_p$ with $p \nmid ab$. Then*

$$\sum_{k=0}^{[\frac{p}{3}]} \binom{3k}{k} \frac{b^{2k}}{a^k} \equiv (-3a)^{-[p/3]} U_{2[\frac{p}{3}]+1}(9b, 3a) \pmod{p}.$$

Proof. Using Lemmas 2.1 and 3.1 we see that for $P, Q \in \mathbb{Z}_p$ with $p \nmid PQ$,

$$\begin{aligned} (3.1) \quad U_{2[\frac{p}{3}]+1}(P, Q) &= \sum_{k=0}^{[p/3]} \binom{\frac{p-r}{3} + k}{\frac{p-r}{3} - k} (-Q)^{\frac{p-r}{3} - k} P^{2k} \\ &\equiv (-Q)^{[\frac{p}{3}]} \sum_{k=0}^{[\frac{p}{3}]} \binom{3k}{k} \frac{1}{27^k} \left(\frac{P^2}{Q}\right)^k \pmod{p}. \end{aligned}$$

Now taking $P = 9b$ and $Q = 3a$ in (3.1) we deduce the result.

Theorem 3.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{[p/3]} \frac{\binom{3k}{k}}{27^k} \equiv \begin{cases} 1 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{9}, \\ -1 \pmod{p} & \text{if } p \equiv \pm 2 \pmod{9}, \\ 0 \pmod{p} & \text{if } p \equiv \pm 4 \pmod{9}. \end{cases}$$

Proof. Taking $a = \frac{1}{3}$ and $b = \frac{1}{9}$ in Theorem 3.1 and then applying Lemma 2.3 we deduce

$$\sum_{k=0}^{[p/3]} \binom{3k}{k} \frac{1}{27^k} \equiv (-1)^{[\frac{p}{3}]} \left(\frac{2[\frac{p}{3}] + 1}{3} \right) \pmod{p}.$$

This yields the result.

Lemma 3.2. *Let $p > 3$ be a prime and $P, Q \in \mathbb{Z}_p$ with $p \nmid PQ$. Then*

$$U_{2[\frac{p}{3}]+1}(P, Q) \equiv \begin{cases} -Q^{1-\frac{p-(\frac{p}{3})}{3}} U_{\frac{p-(\frac{p}{3})}{3}-1}(P, Q) \pmod{p} & \text{if } \left(\frac{P^2-4Q}{p}\right) = 1, \\ -Q^{-\frac{p-(\frac{p}{3})}{3}} U_{\frac{p-(\frac{p}{3})}{3}+1}(P, Q) \pmod{p} & \text{if } \left(\frac{P^2-4Q}{p}\right) = -1. \end{cases}$$

Proof. Since $2[\frac{p}{3}] + 1 = p - \frac{p-(\frac{p}{3})}{3}$ and $\frac{P \pm \sqrt{P^2-4Q}}{2} = \frac{Q}{(P \mp \sqrt{P^2-4Q})/2}$, we see that

$$\begin{aligned} U_{2[\frac{p}{3}]+1}(P, Q) &= \frac{1}{\sqrt{P^2-4Q}} \left\{ \left(\frac{P + \sqrt{P^2-4Q}}{2} \right)^{2[\frac{p}{3}]+1} - \left(\frac{P - \sqrt{P^2-4Q}}{2} \right)^{2[\frac{p}{3}]+1} \right\} \\ &= \frac{1}{\sqrt{P^2-4Q}} \left\{ \left(\frac{P + \sqrt{P^2-4Q}}{2} \right)^p \left(\frac{P - \sqrt{P^2-4Q}}{2} \right)^{\frac{p-(\frac{p}{3})}{3}} \right. \\ &\quad \left. - \left(\frac{P - \sqrt{P^2-4Q}}{2} \right)^p \left(\frac{P + \sqrt{P^2-4Q}}{2} \right)^{\frac{p-(\frac{p}{3})}{3}} \right\}. \end{aligned}$$

Since

$$\begin{aligned} \left(\frac{P \pm \sqrt{P^2-4Q}}{2} \right)^p &\equiv \frac{P^p \pm (\sqrt{P^2-4Q})^p}{2^p} \equiv \frac{P \pm \sqrt{P^2-4Q} (P^2-4Q)^{\frac{p-1}{2}}}{2} \\ &\equiv \frac{P \pm \left(\frac{P^2-4Q}{p}\right) \sqrt{P^2-4Q}}{2} \pmod{p}, \end{aligned}$$

by the above we have

$$\begin{aligned} U_{2[\frac{p}{3}]+1}(P, Q) &\equiv \frac{Q^{-(p-(\frac{p}{3}))/3}}{\sqrt{P^2-4Q}} \left\{ \frac{P + \left(\frac{P^2-4Q}{p}\right) \sqrt{P^2-4Q}}{2} \left(\frac{P - \sqrt{P^2-4Q}}{2} \right)^{\frac{p-(\frac{p}{3})}{3}} \right. \\ &\quad \left. - \frac{P - \left(\frac{P^2-4Q}{p}\right) \sqrt{P^2-4Q}}{2} \left(\frac{P + \sqrt{P^2-4Q}}{2} \right)^{\frac{p-(\frac{p}{3})}{3}} \right\} \pmod{p}. \end{aligned}$$

If $(\frac{P^2-4Q}{p}) = -1$, by the above we have

$$U_{2[\frac{p}{3}]+1}(P, Q) \equiv -Q^{-\frac{p-(\frac{p}{3})}{3}} U_{\frac{p-(\frac{p}{3})}{3}+1}(P, Q) \pmod{p}.$$

If $(\frac{P^2-4Q}{p}) = 1$, by the above and the fact $\frac{P \pm \sqrt{P^2-4Q}}{2} = \frac{Q}{(P \mp \sqrt{P^2-4Q})/2}$ we have

$$U_{2[\frac{p}{3}]+1}(P, Q) \equiv -Q^{1-(p-(\frac{p}{3}))/3} U_{\frac{p-(\frac{p}{3})}{3}-1}(P, Q) \pmod{p}.$$

So the lemma is proved.

Theorem 3.3. *Let $p > 3$ be a prime and $a, b \in \mathbb{Z}_p$ with $p \nmid ab$. Then*

$$\sum_{k=0}^{[p/3]} \binom{3k}{k} \frac{b^{2k}}{a^k} \equiv \begin{cases} (-3a)^{[\frac{p}{3}]+1} U_{\frac{p-(\frac{p}{3})}{3}-1}(9b, 3a) \pmod{p} & \text{if } (\frac{81b^2-12a}{p}) = 1, \\ -(-3a)^{[\frac{p}{3}]} U_{\frac{p-(\frac{p}{3})}{3}+1}(9b, 3a) \pmod{p} & \text{if } (\frac{81b^2-12a}{p}) = -1. \end{cases}$$

Proof. From Theorem 3.1 and Lemma 3.2 we deduce

$$\begin{aligned} \sum_{k=0}^{[p/3]} \binom{3k}{k} \frac{b^{2k}}{a^k} &\equiv (-3a)^{-[\frac{p}{3}]} U_{2[\frac{p}{3}]+1}(9b, 3a) \\ &\equiv \begin{cases} -(-3a)^{-[\frac{p}{3}]} \cdot (3a)^{1-\frac{p-(\frac{p}{3})}{3}} U_{\frac{p-(\frac{p}{3})}{3}-1}(9b, 3a) \pmod{p} & \text{if } (\frac{81b^2-12a}{p}) = 1, \\ -(-3a)^{-[\frac{p}{3}]} \cdot (3a)^{-\frac{p-(\frac{p}{3})}{3}} U_{\frac{p-(\frac{p}{3})}{3}+1}(9b, 3a) \pmod{p} & \text{if } (\frac{81b^2-12a}{p}) = -1. \end{cases} \end{aligned}$$

To see the result we note that $2[\frac{p}{3}] = p-1 - \frac{p-(\frac{p}{3})}{3}$ and so

$$(3a)^{-[\frac{p}{3}] - \frac{p-(\frac{p}{3})}{3}} = (3a)^{[\frac{p}{3}] - (p-1)} \equiv (3a)^{[\frac{p}{3}]} \pmod{p}.$$

Corollary 3.1. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{[p/3]} \frac{\binom{3k}{k}}{(-27)^k} \equiv \begin{cases} F_{\frac{p-(\frac{p}{3})}{3}-1} \pmod{p} & \text{if } (\frac{p}{5}) = 1, \\ -F_{\frac{p-(\frac{p}{3})}{3}+1} \pmod{p} & \text{if } (\frac{p}{5}) = -1. \end{cases}$$

Proof. Taking $a = -\frac{1}{3}$ and $b = \frac{1}{9}$ in Theorem 3.3 we obtain the result.

Theorem 3.4. *Let $p > 5$ be a prime, and let $\varepsilon_p = 1, -1, 0$ according as $p \equiv \pm 1, \pm 2$ or $\pm 4 \pmod{9}$.*

(i) *If $p \equiv 1, 4 \pmod{15}$ and so $p = x^2 + 15y^2$ with $x, y \in \mathbb{Z}$, then*

$$2 \sum_{k=0}^{[p/6]} \frac{\binom{6k}{2k}}{27^{2k}} - \varepsilon_p \equiv \sum_{k=0}^{[p/3]} \frac{\binom{3k}{k}}{(-27)^k} \equiv \begin{cases} 1 \pmod{p} & \text{if } 3 \mid y, \\ (x-5y)/(10y) \pmod{p} & \text{if } 3 \nmid y-x. \end{cases}$$

(ii) If $p \equiv 2, 8 \pmod{15}$ and so $p = 5x^2 + 3y^2$ with $x, y \in \mathbb{Z}$, then

$$2 \sum_{k=0}^{\lfloor p/6 \rfloor} \frac{\binom{6k}{2k}}{27^{2k}} - \varepsilon_p \equiv \sum_{k=0}^{\lfloor p/3 \rfloor} \frac{\binom{3k}{k}}{(-27)^k} \equiv \begin{cases} 1 \pmod{p} & \text{if } 3 \mid y, \\ -(x+y)/(2y) \pmod{p} & \text{if } 3 \nmid y-x. \end{cases}$$

Proof. By Theorem 3.2 we have

$$2 \sum_{k=0}^{\lfloor p/6 \rfloor} \frac{\binom{6k}{2k}}{27^{2k}} = \sum_{k=0}^{\lfloor p/3 \rfloor} \left(\frac{\binom{3k}{k}}{27^k} + \frac{\binom{3k}{k}}{(-27)^k} \right) \equiv \varepsilon_p + \sum_{k=0}^{\lfloor p/3 \rfloor} \frac{\binom{3k}{k}}{(-27)^k} \pmod{p}.$$

If $p = x^2 + 15y^2 \equiv 1, 4 \pmod{15}$, by [6, Theorem 6.2] we have

$$(3.2) \quad F_{\frac{p-1}{3}} \equiv \begin{cases} 0 \pmod{p} & \text{if } 3 \mid y, \\ -x/(5y) \pmod{p} & \text{if } 3 \mid y-x \end{cases} \text{ and } L_{\frac{p-1}{3}} \equiv \begin{cases} 2 \pmod{p} & \text{if } 3 \mid y, \\ -1 \pmod{p} & \text{if } 3 \nmid y. \end{cases}$$

If $p = 5x^2 + 3y^2 \equiv 2, 8 \pmod{15}$, by [6, Theorem 6.2] we have

$$(3.3) \quad F_{\frac{p+1}{3}} \equiv \begin{cases} 0 \pmod{p} & \text{if } 3 \mid y, \\ x/y \pmod{p} & \text{if } 3 \mid y-x \end{cases} \text{ and } L_{\frac{p+1}{3}} \equiv \begin{cases} -2 \pmod{p} & \text{if } 3 \mid y, \\ 1 \pmod{p} & \text{if } 3 \nmid y. \end{cases}$$

Note that $2F_{n \pm 1} = L_n \pm F_n$. From Corollary 3.1 and the above we deduce the result.

Theorem 3.5. Let p be an odd prime with $p \equiv 1, 2, 4, 8 \pmod{15}$.

(i) If $p \equiv 1, 4 \pmod{15}$ and so $p = x^2 + 15y^2$ with $x, y \in \mathbb{Z}$, then

$$\sum_{k=0}^{\lfloor p/3 \rfloor} \binom{3k}{k} \frac{1}{3^k} \equiv \begin{cases} 1 \pmod{p} & \text{if } 3 \mid y, \\ -(3x+5y)/(10y) \pmod{p} & \text{if } 3 \mid y-x. \end{cases}$$

(ii) If $p \equiv 2, 8 \pmod{15}$ and so $p = 5x^2 + 3y^2$ with $x, y \in \mathbb{Z}$, then

$$\sum_{k=0}^{\lfloor p/3 \rfloor} \binom{3k}{k} \frac{1}{3^k} \equiv \begin{cases} 1 \pmod{p} & \text{if } 3 \mid y, \\ (3x-y)/(2y) \pmod{p} & \text{if } 3 \mid y-x. \end{cases}$$

Proof. It is known that $U_n(3, 1) = F_{2n} = F_n L_n$. Thus, putting $a = b = \frac{1}{3}$ in Theorem 3.3 we see that

$$\sum_{k=0}^{\lfloor p/3 \rfloor} \frac{\binom{3k}{k}}{3^k} \equiv \begin{cases} -U_{\frac{p-1}{3}-1}(3, 1) = -F_{\frac{p-1}{3}-1} L_{\frac{p-1}{3}-1} \pmod{p} & \text{if } p \equiv 1 \pmod{3}, \\ U_{\frac{p+1}{3}+1}(3, 1) = F_{\frac{p+1}{3}+1} L_{\frac{p+1}{3}+1} \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

It is easily seen that

$$2F_{n \pm 1} = L_n \pm F_n \quad \text{and} \quad 2L_{n \pm 1} = 5F_n \pm L_n.$$

Thus, if $p = x^2 + 15y^2 \equiv 1, 4 \pmod{15}$, using (3.2) we see that

$$F_{\frac{p-1}{3}-1} L_{\frac{p-1}{3}-1} = \frac{1}{4} (L_{\frac{p-1}{3}} - F_{\frac{p-1}{3}}) (5F_{\frac{p-1}{3}} - L_{\frac{p-1}{3}}) \equiv \begin{cases} -1 \pmod{p} & \text{if } 3 \mid y, \\ (3x+5y)/(10y) \pmod{p} & \text{if } 3 \mid y-x. \end{cases}$$

If $p = 5x^2 + 3y^2 \equiv 2, 8 \pmod{15}$, using (3.3) we see that

$$F_{\frac{p+1}{3}+1} L_{\frac{p+1}{3}+1} = \frac{1}{4} (L_{\frac{p+1}{3}} + F_{\frac{p+1}{3}}) (5F_{\frac{p+1}{3}} + L_{\frac{p+1}{3}}) \equiv \begin{cases} 1 \pmod{p} & \text{if } 3 \mid y, \\ (3x-y)/(2y) \pmod{p} & \text{if } 3 \mid y-x. \end{cases}$$

Now combining all the above we obtain the result.

Theorem 3.6. *Let p be an odd prime with $(\frac{p}{13}) = (\frac{p}{3})$. Then*

$$\sum_{k=0}^{[p/3]} \binom{3k}{k} \frac{1}{(-3)^k} \equiv \begin{cases} 1 \pmod{p} & \text{if } p = x^2 + xy + 88y^2, 10x^2 + 7xy + 10y^2, \\ & \text{or if } p = 11x^2 + xy + 8y^2, \\ -(25x + 10y)/(13y) \pmod{p} & \text{if } p = 25x^2 + 7xy + 4y^2, \\ (43x + 12y)/(13y) \pmod{p} & \text{if } p = 43x^2 + 37xy + 10y^2 \neq 43, \\ -(5x + 8y)/(13y) \pmod{p} & \text{if } p = 5x^2 + 3xy + 18y^2 \neq 5, \\ -(47x + 9y)/(13y) \pmod{p} & \text{if } p = 47x^2 + 5xy + 2y^2 \neq 47. \end{cases}$$

Proof. Taking $b = \frac{1}{3}$ and $a = -\frac{1}{3}$ in Theorem 3.3 and applying (2.2) we see that

$$\sum_{k=0}^{[p/3]} \binom{3k}{k} \frac{1}{(-3)^k} \equiv \begin{cases} U_{\frac{p-1}{3}-1}(3, -1) = -\frac{1}{2}(3U_{\frac{p-1}{3}}(3, -1) - V_{\frac{p-1}{3}}(3, -1)) \pmod{p} & \text{if } (\frac{13}{p}) = 1, \\ -U_{\frac{p+1}{3}+1}(3, -1) = -\frac{1}{2}(3U_{\frac{p+1}{3}}(3, -1) + V_{\frac{p+1}{3}}(3, -1)) \pmod{p} & \text{if } (\frac{13}{p}) = -1. \end{cases}$$

Now applying [9, Corollary 6.7] we deduce the result.

Theorem 3.7. *Let p be an odd prime with $(\frac{p}{3})(\frac{p}{5})(\frac{p}{17}) = 1$. Then*

$$\sum_{k=0}^{[p/3]} \binom{3k}{k} (-3)^k \equiv \begin{cases} 1 \pmod{p} & \text{if } p = x^2 + xy + 64y^2, 3x^2 + 3xy + 22y^2, \\ & \text{or if } p = 8x^2 + xy + 8y^2, 5x^2 + 5xy + 14y^2, \\ -(171x + 74y)/(85y) \pmod{p} & \text{if } p = 19x^2 + 7xy + 4y^2 \neq 19, \\ -(63x + 65y)/(85y) \pmod{p} & \text{if } p = 7x^2 + 5xy + 10y^2 \neq 7, \\ -(63x + 13y)/(17y) \pmod{p} & \text{if } p = 35x^2 + 5xy + 2y^2, \\ (99x - 29y)/(17y) \pmod{p} & \text{if } p = 11x^2 + 3xy + 6y^2 \neq 11. \end{cases}$$

Proof. Taking $b = 1$ and $a = -\frac{1}{3}$ in Theorem 3.3 and applying (2.2) we see that

$$\sum_{k=0}^{[p/3]} \binom{3k}{k} (-3)^k \equiv \begin{cases} U_{\frac{p-1}{3}-1}(9, -1) = \frac{1}{2}(V_{\frac{p-1}{3}}(9, -1) - 9U_{\frac{p-1}{3}}(9, -1)) \pmod{p} & \text{if } (\frac{85}{p}) = 1, \\ -U_{\frac{p+1}{3}+1}(9, -1) = -\frac{1}{2}(V_{\frac{p+1}{3}}(9, -1) + 9U_{\frac{p+1}{3}}(9, -1)) \pmod{p} & \text{if } (\frac{85}{p}) = -1. \end{cases}$$

Now applying [9, Corollary 6.9] we deduce the result.

Let (u, v) be the greatest common divisor of integers u and v . For $a, b, c \in \mathbb{Z}$ we use $[a, b, c]$ to denote the equivalence class containing the form $ax^2 + bxy + cy^2$. It is well known that

$$(3.4) \quad [a, b, c] = [c, -b, a] = [a, 2ak + b, ak^2 + bk + c] \quad \text{for } k \in \mathbb{Z}.$$

We also use $H(d)$ to denote the form class group of discriminant d . Let $\omega = (-1 + \sqrt{-3})/2$. Following [6] and [9] we use $(\frac{a+b\omega}{m})_3$ ($3 \nmid m$) to denote the cubic Jacobi symbol. For a prime $p > 3$ and $k \in \mathbb{Z}_p$ with $k^2 + 3 \not\equiv 0 \pmod{p}$, using [6, Corollary 6.1] we can easily determine $(\frac{k+1+2\omega}{p})_3$. In particular, by [6, Proposition 2.1] we have $(\frac{1+2\omega}{p})_3 = 1$.

For later convenience, following [9] we introduce the following notation.

Definition 3.1. Suppose $u, v, d \in \mathbb{Z}$, $dv(u^2 - dv^2) \neq 0$ and $(u, v) = 1$. Let $u^2 - dv^2 = 2^\alpha 3^r W(2 \nmid W, 3 \nmid W)$ and let w be the product of all distinct prime divisors of W . Define

$$k_2(u, v, d) = \begin{cases} 2 & \text{if } d \equiv 2, 3 \pmod{4}, \\ 2 & \text{if } d \equiv 1 \pmod{8}, \alpha > 0 \text{ and } \alpha \equiv 0, 1 \pmod{3}, \\ 1 & \text{otherwise,} \end{cases}$$

$$k_3(u, v, d) = \begin{cases} 3^{\text{ord}_3 v + 1} & \text{if } 3 \mid r \text{ and } 3 \nmid u, \\ 9 & \text{if } 3 \nmid r \text{ and } 3 \nmid u, \\ 3 & \text{if } 3 \nmid r - 2, 3 \mid u \text{ and } 9 \nmid u, \\ 1 & \text{otherwise} \end{cases}$$

and $k(u, v, d) = k_2(u, v, d)k_3(u, v, d)w/(u, w)$.

Lemma 3.3 ([9, Theorem 6.1 and Remark 6.1]). Let $p > 3$ be a prime, and $P, Q \in \mathbb{Z}$ with $p \nmid Q$ and $(\frac{-3(P^2 - 4Q)}{p}) = 1$. Assume $P^2 - 4Q = df^2$ ($d, f \in \mathbb{Z}$) and $p = ax^2 + bxy + cy^2$ with $a, b, c, x, y \in \mathbb{Z}$, $(a, 6p \cdot 4Q/(P, f)^2) = 1$ and $b^2 - 4ac = -3k^2d$, where $k = k(P/(P, f), f/(P, f), d)$. Then

$$U_{(p - (\frac{p}{3}))/3}(P, Q) \equiv \begin{cases} 0 \pmod{p} & \text{if } (\frac{\frac{bf}{(P, f)} - \frac{kP}{(P, f)}(1+2\omega)}{a})_3 = 1, \\ -\frac{2ax+by}{kdfy}(\frac{-Q}{p})(-Q)^{\frac{p - (\frac{p}{3})}{6}} \pmod{p} & \text{if } (\frac{\frac{bf}{(P, f)} - \frac{kP}{(P, f)}(1+2\omega)}{a})_3 = \omega, \\ \frac{2ax+by}{kdfy}(\frac{-Q}{p})(-Q)^{\frac{p - (\frac{p}{3})}{6}} \pmod{p} & \text{if } (\frac{\frac{bf}{(P, f)} - \frac{kP}{(P, f)}(1+2\omega)}{a})_3 = \omega^2 \end{cases}$$

and

$$V_{(p - (\frac{p}{3}))/3}(P, Q) \equiv \begin{cases} 2(\frac{p}{3})(\frac{-Q}{p})(-Q)^{\frac{p - (\frac{p}{3})}{6}} \pmod{p} & \text{if } (\frac{\frac{bf}{(P, f)} - \frac{kP}{(P, f)}(1+2\omega)}{a})_3 = 1, \\ -(\frac{p}{3})(\frac{-Q}{p})(-Q)^{\frac{p - (\frac{p}{3})}{6}} \pmod{p} & \text{if } (\frac{\frac{bf}{(P, f)} - \frac{kP}{(P, f)}(1+2\omega)}{a})_3 \neq 1. \end{cases}$$

Moreover, the criteria for $p \mid U_{(p - (\frac{p}{3}))/3}(P, Q)$ and $V_{(p - (\frac{p}{3}))/3}(P, Q) \pmod{p}$ are also true when $p = a$.

Theorem 3.8. *Let $p > 3$ be a prime with $(\frac{p}{23}) = 1$. Then*

$$\sum_{k=0}^{[p/3]} \binom{3k}{k} \equiv \begin{cases} 1 \pmod{p} & \text{if } p = x^2 + xy + 52y^2, \ 8x^2 + 7xy + 8y^2, \\ (39x - 10y)/(23y) \pmod{p} & \text{if } p = 13x^2 + xy + 4y^2 \neq 13, \\ -(87x + 19y)/(23y) \pmod{p} & \text{if } p = 29x^2 + 5xy + 2y^2 \neq 29. \end{cases}$$

Proof. Putting $a = b = 1$ in Theorem 3.3 and applying (2.2) we see that

$$\sum_{k=0}^{[p/3]} \binom{3k}{k} \equiv \begin{cases} (-3)^{\frac{p-1}{3}+1} \cdot \frac{1}{6}(9U_{\frac{p-1}{3}}(9, 3) - V_{\frac{p-1}{3}}(9, 3)) \pmod{p} & \text{if } 3 \mid p-1, \\ -(-3)^{\frac{p-2}{3}} \cdot \frac{1}{2}(9U_{\frac{p+1}{3}}(9, 3) + V_{\frac{p+1}{3}}(9, 3)) \pmod{p} & \text{if } 3 \mid p-2. \end{cases}$$

Since $(\frac{p}{23}) = 1$ we have $(\frac{-3 \cdot 69}{p}) = (\frac{-23}{p}) = (\frac{p}{23}) = 1$ and $(\frac{69}{p}) = (\frac{p}{3})$. Thus p is represented by some class in $H(-207)$. It is known that

$$H(-207) = \{[1, 1, 52], [8, 7, 8], [4, 1, 13], [4, -1, 13], [2, 1, 26], [2, -1, 26]\}.$$

From (3.4) one can easily see that $[2, -1, 26] = [2, -5, 29] = [29, 5, 2]$ and $[8, 7, 8] = [8, 23, 23] = [23, -23, 8]$. Note that

$$\begin{aligned} \left(\frac{1-9(1+2\omega)}{1}\right)_3 &= 1, & \left(\frac{-23-9(1+2\omega)}{23}\right)_3 &= \left(\frac{1+2\omega}{23}\right)_3 = 1, \\ \left(\frac{1-9(1+2\omega)}{13}\right)_3 &= \left(\frac{-3+1+2\omega}{13}\right)_3 = \omega, \\ \left(\frac{5-9(1+2\omega)}{29}\right)_3 &= \left(\frac{-7+1+2\omega}{29}\right)_3 = \omega^2. \end{aligned}$$

Since $k(9, 1, 69) = 1$ by Definition 3.1, putting $P = 9$, $Q = 3$, $d = 69$, $f = 1$ and $k = 1$ in Lemma 3.3 and applying the above we see that

$$U_{\frac{p-(\frac{p}{3})}{3}}(9, 3) \equiv \begin{cases} 0 \pmod{p} & \text{if } p = x^2 + xy + 52y^2 \\ & \text{or } p = 8x^2 + 7xy + 8y^2, \\ -\frac{26x+y}{69y}(-3)^{\frac{p-1}{6}} \pmod{p} & \text{if } p = 13x^2 + xy + 4y^2 \neq 13, \\ -\frac{58x+5y}{69y}(-3)^{\frac{p+1}{6}} \pmod{p} & \text{if } p = 29x^2 + 5xy + 2y^2 \neq 29 \end{cases}$$

and

$$V_{\frac{p-(\frac{p}{3})}{3}}(9, 3) \equiv \begin{cases} 2(-3)^{(p-1)/6} \pmod{p} & \text{if } p = x^2 + xy + 52y^2 \\ 2(-3)^{(p+1)/6} \pmod{p} & \text{if } p = 8x^2 + 7xy + 8y^2, \\ -(-3)^{(p-1)/6} \pmod{p} & \text{if } p = 13x^2 + xy + 4y^2, \\ -(-3)^{(p+1)/6} \pmod{p} & \text{if } p = 29x^2 + 5xy + 2y^2. \end{cases}$$

Now combining all the above with the fact $(-3)^{(p-1)/2} \equiv (\frac{-3}{p}) = (\frac{p}{3}) \pmod{p}$ we deduce the result.

Theorem 3.9. *Let $p > 3$ be a prime with $(\frac{p}{31}) = 1$. Then*

$$\sum_{k=0}^{[p/3]} \binom{3k}{k} (-1)^k \equiv \begin{cases} 1 \pmod{p} & \text{if } p = x^2 + xy + 70y^2, 9x^2 + 9xy + 10y^2 \\ & \text{or } p = 8x^2 + 3xy + 9y^2, \\ (15x - 14y)/(31y) \pmod{p} & \text{if } p = 5x^2 + xy + 14y^2 \neq 5, \\ (21x - 14y)/(31y) \pmod{p} & \text{if } p = 7x^2 + xy + 10y^2 \neq 7, \\ (57x - 8y)/(31y) \pmod{p} & \text{if } p = 19x^2 + 5xy + 4y^2 \neq 19, \\ -(105x + 17y)/(31y) \pmod{p} & \text{if } p = 35x^2 + xy + 2y^2. \end{cases}$$

Proof. Putting $a = -1$ and $b = 1$ in Theorem 3.3 and applying (2.2) we see that

$$\sum_{k=0}^{[p/3]} \binom{3k}{k} (-1)^k \equiv \begin{cases} 3^{\frac{p-1}{3}+1} \cdot \frac{1}{(-6)} (9U_{\frac{p-1}{3}}(9, -3) - V_{\frac{p-1}{3}}(9, -3)) \pmod{p} & \text{if } 3 \mid p-1, \\ -3^{\frac{p-2}{3}} \cdot \frac{1}{2} (9U_{\frac{p+1}{3}}(9, -3) + V_{\frac{p+1}{3}}(9, -3)) \pmod{p} & \text{if } 3 \nmid p-1. \end{cases}$$

Since $(\frac{p}{31}) = 1$ we have $(\frac{-3 \cdot 93}{p}) = (\frac{-31}{p}) = (\frac{p}{31}) = 1$ and $(\frac{93}{p}) = (\frac{p}{3})$. Thus p is represented by some class in $H(-279)$. It is known that

$$H(-279) = \{[1, 1, 70], [9, 9, 10], [2, 1, 35], [2, -1, 35], [5, 1, 14], [5, -1, 14], [7, 1, 10], [7, -1, 10], [4, 3, 18], [4, -3, 18], [8, 3, 9], [8, -3, 9]\}.$$

One can easily see that $[2, -1, 35] = [35, 1, 2]$, $[4, 3, 18] = [4, -5, 19] = [19, 5, 4]$, $[8, 3, 9] = [8, -29, 35] = [35, 29, 8]$ and $[9, 9, 10] = [10, -9, 9] = [10, 31, 31] = [31, -31, 10]$. Note that

$$\begin{aligned} \left(\frac{1-9(1+2\omega)}{1}\right)_3 &= 1, & \left(\frac{-31-9(1+2\omega)}{31}\right)_3 &= \left(\frac{1+2\omega}{31}\right)_3 = 1, \\ \left(\frac{29-9(1+2\omega)}{35}\right)_3 &= \left(\frac{24+1+2\omega}{5}\right)_3 \left(\frac{24+1+2\omega}{7}\right)_3 = \omega^2 \cdot \omega = 1, \\ \left(\frac{1-9(1+2\omega)}{5}\right)_3 &= \left(\frac{1+1+2\omega}{5}\right)_3 = \omega, & \left(\frac{1-9(1+2\omega)}{7}\right)_3 &= \left(\frac{3+1+2\omega}{7}\right)_3 = \omega, \\ \left(\frac{5-9(1+2\omega)}{19}\right)_3 &= \left(\frac{-9+1+2\omega}{19}\right)_3 = \omega, \\ \left(\frac{1-9(1+2\omega)}{35}\right)_3 &= \left(\frac{-4+1+2\omega}{5}\right)_3 \left(\frac{-4+1+2\omega}{7}\right)_3 = \omega \cdot \omega = \omega^2. \end{aligned}$$

Since $k(9, 1, 93) = 1$ by Definition 3.1, putting $P = 9$, $Q = -3$, $d = 93$, $f = 1$ and $k = 1$ in Lemma 3.3 and applying the above we see that

$$U_{\frac{p-(\frac{p}{3})}{3}}(9, -3) \equiv \begin{cases} 0 \pmod{p} & \text{if } p = x^2 + xy + 70y^2, 9x^2 + 9xy + 10y^2 \\ & \text{or } p = 8x^2 + 3xy + 9y^2, \\ -\frac{10x+y}{93y} \left(\frac{3}{p}\right) 3^{\frac{p+1}{6}} \pmod{p} & \text{if } p = 5x^2 + xy + 14y^2 \neq 5, \\ -\frac{14x+y}{93y} \left(\frac{3}{p}\right) 3^{\frac{p-1}{6}} \pmod{p} & \text{if } p = 7x^2 + xy + 10y^2 \neq 7, \\ -\frac{38x+5y}{93y} \left(\frac{3}{p}\right) 3^{\frac{p-1}{6}} \pmod{p} & \text{if } p = 19x^2 + 5xy + 4y^2 \neq 19, \\ \frac{70x+y}{93y} \left(\frac{3}{p}\right) 3^{\frac{p+1}{6}} \pmod{p} & \text{if } p = 35x^2 + xy + 2y^2 \end{cases}$$

and

$$V_{\frac{p-(\frac{p}{3})}{3}}(9, -3) \equiv \begin{cases} 2(\frac{3}{p})3^{(p-1)/6} \pmod{p} & \text{if } p = x^2 + xy + 70y^2, 9x^2 + 9xy + 10y^2, \\ -2(\frac{3}{p})3^{(p+1)/6} \pmod{p} & \text{if } p = 8x^2 + 3xy + 9y^2, \\ (\frac{3}{p})3^{(p+1)/6} \pmod{p} & \text{if } p = 5x^2 + xy + 14y^2, 35x^2 + xy + 2y^2, \\ -(\frac{3}{p})3^{(p-1)/6} \pmod{p} & \text{if } p = 7x^2 + xy + 10y^2, 19x^2 + 5xy + 4y^2. \end{cases}$$

Now combining all the above we deduce the result.

Theorem 3.10. *Let $p > 3$ be a prime and $a \in \mathbb{Z}_p$ with $(\frac{a(4-27a)}{p}) = -1$. Then $x \equiv \sum_{k=0}^{[p/3]} \binom{3k}{k} a^k \pmod{p}$ is the unique solution of the cubic congruence $(27a-4)x^3 + 3x + 1 \equiv 0 \pmod{p}$.*

Proof. As $(\frac{a(4-27a)}{p}) = -1$ we have $(\frac{81a^2-12a}{p}) = (\frac{-3}{p})(\frac{a(4-27a)}{p}) = -(\frac{-3}{p}) = -(\frac{p}{3})$. Thus putting $b = a$ in Theorem 3.3 we have

$$\sum_{k=0}^{[p/3]} \binom{3k}{k} a^k \equiv \begin{cases} -(3a)^{\frac{p-1}{3}} U_{\frac{p-1}{3}+1}(9a, 3a) \pmod{p} & \text{if } p \equiv 1 \pmod{3}, \\ (-3a)^{\frac{p-2}{3}+1} U_{\frac{p+1}{3}-1}(9a, 3a) \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

From [8, Theorem 2.1] or [9, Remark 6.1] we know that

$$U_{\frac{p-(\frac{p}{3})}{3}}(9a, 3a) \equiv \frac{1}{27a-4} \left(\frac{-a}{p} \right) (3a)^{\frac{p-(\frac{p}{3})}{6}-1} (-3x^2 + 2x + 18a) \pmod{p}$$

and

$$V_{\frac{p-(\frac{p}{3})}{3}}(9a, 3a) \equiv \left(\frac{3a}{p} \right) (3a)^{\frac{p-(\frac{p}{3})}{6}-1} (x^2 - 6a) \pmod{p},$$

where x is the unique solution of the congruence $X^3 - 9aX - 27a^2 \equiv 0 \pmod{p}$. Hence

$$\begin{aligned} & 9aU_{\frac{p-(\frac{p}{3})}{3}}(9a, 3a) + \left(\frac{p}{3} \right) V_{\frac{p-(\frac{p}{3})}{3}}(9a, 3a) \\ & \equiv \frac{1}{27a-4} \left(\frac{-a}{p} \right) (3a)^{\frac{p-(\frac{p}{3})}{6}-1} (9a(-3x^2 + 2x + 18a) + (27a-4)(x^2 - 6a)) \\ & = -\frac{2}{27a-4} \left(\frac{-a}{p} \right) (3a)^{\frac{p-(\frac{p}{3})}{6}-1} (2x^2 - 9ax - 12a) \pmod{p}. \end{aligned}$$

Now putting $b = a$ in Theorem 3.3 and applying (2.2) and the above we deduce

$$\begin{aligned} \sum_{k=0}^{[p/3]} \binom{3k}{k} a^k & \equiv -\left(\frac{p}{3} \right) (3a)^{\frac{p-(\frac{p}{3})}{3}} U_{\frac{p-(\frac{p}{3})}{3}+(\frac{p}{3})}(9a, 3a) \\ & = -\left(\frac{p}{3} \right) (3a)^{\frac{p-(\frac{p}{3})}{3}} \cdot \frac{1}{2(3a)^{(1-(\frac{p}{3}))/2}} \left(9aU_{\frac{p-(\frac{p}{3})}{3}}(9a, 3a) + \left(\frac{p}{3} \right) V_{\frac{p-(\frac{p}{3})}{3}}(9a, 3a) \right) \\ & \equiv \left(\frac{p}{3} \right) (3a)^{\frac{p-(\frac{p}{3})}{3}+\frac{(\frac{p}{3})-1}{2}} \frac{1}{27a-4} \left(\frac{-a}{p} \right) (3a)^{\frac{p-(\frac{p}{3})}{6}-1} (2x^2 - 9ax - 12a) \\ & \equiv \frac{1}{3a(27a-4)} (2x^2 - 9ax - 12a) \pmod{p}. \end{aligned}$$

As $x^3 \equiv 9ax + 27a^2 \pmod{p}$ we see that

$$(2x^2 - 9ax - 12a)(2x + 9a) \equiv 3a(4 - 27a)x \pmod{p}.$$

Hence

$$\sum_{k=0}^{\lfloor p/3 \rfloor} \binom{3k}{k} a^k \equiv \frac{1}{3a(27a-4)} (2x^2 - 9ax - 12a) \equiv -\frac{x}{2x+9a} \pmod{p}.$$

Set $x_0 = -\frac{x}{2x+9a}$. Then $x = -\frac{9ax_0}{2x_0+1}$ and

$$\begin{aligned} & (27a-4)x_0^3 + 3x_0 + 1 \\ &= (4-27a) \frac{x^3}{(2x+9a)^3} - \frac{3x}{2x+9a} + 1 = -\frac{27a(x^3 - 9ax - 27a^2)}{(2x+9a)^3} \equiv 0 \pmod{p}. \end{aligned}$$

So the theorem is proved.

4. Congruences via combinatorial sums.

For $m, n \in \mathbb{N}$ and $r \in \mathbb{Z}$ let

$$T_{r(m)}^n = \sum_{\substack{k \in \{0,1,\dots,n\} \\ k \equiv r \pmod{m}}} \binom{n}{k}.$$

From [3, Corollary 1.8] or [11] we know that

$$(4.1) \quad T_{r(m)}^n = T_{n-r(m)}^n \quad \text{and} \quad T_{r(m)}^{n+1} = T_{r(m)}^n + T_{r-1(m)}^n.$$

Theorem 4.1. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{\lfloor \frac{p-1}{6} \rfloor} \frac{\binom{6k}{3k}}{(-64)^k} \equiv \begin{cases} \frac{1}{3}((-1)^{\lfloor \frac{p+1}{4} \rfloor} + 2(-1)^{\frac{p-1}{2}}) \pmod{p} & \text{if } 3 \mid p-1, \\ \frac{1}{3}((-1)^{\lfloor \frac{p+1}{4} \rfloor} - (-1)^{\frac{p-1}{2}}) \pmod{p} & \text{if } 3 \nmid p-1. \end{cases}$$

Proof. It is known that (see for example ([1, (1.56)], [3, Theorem 1.1])

$$(4.2) \quad T_{0(3)}^n = \sum_{k=0}^{\lfloor n/3 \rfloor} \binom{n}{3k} = \begin{cases} \frac{1}{3}(2^n + 2(-1)^n) & \text{if } 3 \mid n, \\ \frac{1}{3}(2^n - (-1)^n) & \text{if } 3 \nmid n. \end{cases}$$

Thus, taking $n = \frac{p-1}{2}$ and using Lemma 2.3 we see that

$$\sum_{k=0}^{\lfloor \frac{p-1}{6} \rfloor} \frac{\binom{6k}{3k}}{(-64)^k} \equiv \sum_{k=0}^{\lfloor \frac{p-1}{6} \rfloor} \binom{\frac{p-1}{2}}{3k} = \begin{cases} \frac{1}{3}(2^{\frac{p-1}{2}} + 2(-1)^{\frac{p-1}{2}}) \pmod{p} & \text{if } 3 \mid p-1, \\ \frac{1}{3}(2^{\frac{p-1}{2}} - (-1)^{\frac{p-1}{2}}) \pmod{p} & \text{if } 3 \nmid p-1. \end{cases}$$

To see the result, we note that $2^{\frac{p-1}{2}} \equiv (-1)^{\lfloor \frac{p+1}{4} \rfloor} \pmod{p}$.

Theorem 4.2. *Let p be an odd prime. Then*

$$\sum_{k=0}^{\lfloor \frac{p-1}{8} \rfloor} \frac{\binom{8k}{4k}}{4^{4k-1}} \equiv \begin{cases} 1 + (-1)^{\frac{p-1}{8}} 2^{\frac{p+3}{4}} \pmod{p} & \text{if } p \equiv 1 \pmod{8}, \\ -1 + (-1)^{\frac{p-3}{8}} 2^{\frac{p+1}{4}} \pmod{p} & \text{if } p \equiv 3 \pmod{8}, \\ -1 \pmod{p} & \text{if } p \equiv 5 \pmod{8}, \\ 1 - (-1)^{\frac{p-7}{8}} 2^{\frac{p+1}{4}} \pmod{p} & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Proof. It is known that (see for example ([1, (1.58)], [3, Theorem 1.2])

$$(4.3) \quad T_{0(4)}^n = \sum_{k=0}^{\lfloor n/4 \rfloor} \binom{n}{4k} = \begin{cases} \frac{1}{2}(2^{n-1} + (-1)^{\lfloor n/4 \rfloor} 2^{\lfloor n/2 \rfloor}) & \text{if } n \equiv 0, 1 \pmod{4}, \\ 2^{n-2} & \text{if } n \equiv 2 \pmod{4}, \\ \frac{1}{2}(2^{n-1} - (-1)^{\lfloor n/4 \rfloor} 2^{\lfloor n/2 \rfloor}) & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Thus, taking $n = \frac{p-1}{2}$ and using Lemma 2.3 we see that

$$\sum_{k=0}^{\lfloor \frac{p-1}{8} \rfloor} \frac{\binom{8k}{4k}}{(-4)^{4k}} \equiv \sum_{k=0}^{\lfloor \frac{p-1}{8} \rfloor} \binom{\frac{p-1}{2}}{4k} = \begin{cases} \frac{1}{2}(2^{\frac{p-1}{2}-1} + (-1)^{\frac{p-1}{8}} 2^{\frac{p-1}{4}}) \pmod{p} & \text{if } 8 \mid p-1, \\ \frac{1}{2}(2^{\frac{p-1}{2}-1} + (-1)^{\frac{p-3}{8}} 2^{\frac{p-3}{4}}) \pmod{p} & \text{if } 8 \mid p-3, \\ 2^{\frac{p-1}{2}-2} \pmod{p} & \text{if } 8 \mid p-5, \\ \frac{1}{2}(2^{\frac{p-1}{2}-1} - (-1)^{\frac{p-7}{8}} 2^{\frac{p-3}{4}}) \pmod{p} & \text{if } 8 \mid p-7. \end{cases}$$

To see the result, we note that $2^{\frac{p-1}{2}} \equiv (-1)^{\lfloor \frac{p+1}{4} \rfloor} \pmod{p}$.

Theorem 4.3. *Let $p > 3$ be a prime. Then*

$$6 \sum_{k=0}^{\lfloor \frac{p-1}{12} \rfloor} \frac{\binom{12k}{6k}}{4^{6k}} \equiv \begin{cases} 2 \cdot 3^{(p-1)/4} + 3 \pmod{p} & \text{if } p \equiv 1 \pmod{24}, \\ 3^{(p-1)/4} - 2 \pmod{p} & \text{if } p \equiv 5 \pmod{24}, \\ -2 \pmod{p} & \text{if } p \equiv 7 \pmod{24}, \\ -3^{(p+1)/4} \pmod{p} & \text{if } p \equiv 11 \pmod{24}, \\ 1 - 2 \cdot 3^{(p-1)/4} \pmod{p} & \text{if } p \equiv 13 \pmod{24}, \\ -3^{(p-1)/4} \pmod{p} & \text{if } p \equiv 17 \pmod{24}, \\ -3 \pmod{p} & \text{if } p \equiv 19 \pmod{24}, \\ 2 + 3^{(p+1)/4} \pmod{p} & \text{if } p \equiv 23 \pmod{24}. \end{cases}$$

Proof. From [3, Theorem 1.9] we know that

$$(4.4) \quad 6 \sum_{k=0}^{\lfloor n/6 \rfloor} \binom{n}{6k} - 2^n = \begin{cases} 3^{(n+1)/2} + 1 & \text{if } n \equiv \pm 1 \pmod{12}, \\ -2 & \text{if } n \equiv \pm 3 \pmod{12}, \\ -3^{(n+1)/2} + 1 & \text{if } n \equiv \pm 5 \pmod{12}, \\ 2(3^{n/2} + 1) & \text{if } n \equiv 0 \pmod{12}, \\ 3^{n/2} - 1 & \text{if } n \equiv \pm 2 \pmod{12}, \\ -3^{n/2} - 1 & \text{if } n \equiv \pm 4 \pmod{12}, \\ 2(1 - 3^{n/2}) & \text{if } n \equiv 6 \pmod{12}. \end{cases}$$

Thus, by the above and Lemma 2.3 we have

$$\begin{aligned}
& 6 \sum_{k=0}^{\lfloor \frac{p-1}{12} \rfloor} \frac{\binom{12k}{6k}}{(-4)^{6k}} - (-1)^{\lfloor \frac{p+1}{4} \rfloor} \\
& \equiv 6 \sum_{k=0}^{\lfloor \frac{p-1}{12} \rfloor} \binom{\frac{p-1}{2}}{6k} - 2^{\frac{p-1}{2}} = \begin{cases} 3^{(p+1)/4} + 1 \pmod{p} & \text{if } p \equiv 23 \pmod{24}, \\ -2 \pmod{p} & \text{if } p \equiv 7, 19 \pmod{24}, \\ -3^{(p+1)/4} + 1 \pmod{p} & \text{if } p \equiv 11 \pmod{24}, \\ 2(3^{(p-1)/4} + 1) \pmod{p} & \text{if } p \equiv 1 \pmod{24}, \\ 3^{(p-1)/4} - 1 \pmod{p} & \text{if } p \equiv 5 \pmod{24}, \\ -3^{(p-1)/4} - 1 \pmod{p} & \text{if } p \equiv 17 \pmod{24}, \\ 2(1 - 3^{(p-1)/4}) \pmod{p} & \text{if } p \equiv 13 \pmod{24}. \end{cases}
\end{aligned}$$

This yields the result.

Theorem 4.4. *Let p be a prime greater than 5. Then*

$$5 \sum_{k=0}^{\lfloor \frac{p-1}{10} \rfloor} \frac{\binom{10k}{5k}}{(-4)^{5k}} - (-1)^{\lfloor \frac{p+1}{4} \rfloor} \equiv \begin{cases} 4 \cdot 5^{\frac{p-1}{4}} \pmod{p} & \text{if } p \equiv 1 \pmod{20}, \\ 2 \cdot 5^{\frac{p+1}{4}} \pmod{p} & \text{if } p \equiv 3 \pmod{20}, \\ 5^{\frac{p+1}{4}} \pmod{p} & \text{if } p \equiv 7 \pmod{20}, \\ 5^{\frac{p-1}{4}} \pmod{p} & \text{if } p \equiv 9 \pmod{20}, \\ -5^{\frac{p+1}{4}} \pmod{p} & \text{if } p \equiv 11 \pmod{20}, \\ -2 \cdot 5^{\frac{p-1}{4}} \pmod{p} & \text{if } p \equiv 13 \pmod{20}, \\ 3 \cdot 5^{\frac{p-1}{4}} \pmod{p} & \text{if } p \equiv 17 \pmod{20}, \\ -5^{\frac{p+1}{4}} \pmod{p} & \text{if } p \equiv 19 \pmod{20}. \end{cases}$$

Proof. Let

$$\Delta_5(r, n) = \begin{cases} 5T_{\frac{n-1}{2}+r(5)}^n - 2^n & \text{if } 2 \nmid n, \\ 5T_{\frac{n}{2}+r(5)}^n - 2^n & \text{if } 2 \mid n. \end{cases}$$

From [3, Theorem 1.6] we know that

$$\Delta_5(0, n) = 2(-1)^n L_n, \quad \Delta_5(\pm 1, n) = (-1)^n L_{n-1}, \quad \Delta_5(\pm 2, n) = (-1)^{n+1} L_{n+1}.$$

From this we deduce

$$(4.5) \quad 5 \sum_{5 \mid k} \binom{\frac{p-1}{2}}{k} - 2^{\frac{p-1}{2}} = \begin{cases} 2L_{\frac{p-1}{2}} & \text{if } p \equiv 1 \pmod{20}, \\ -2L_{\frac{p-1}{2}} & \text{if } p \equiv 3 \pmod{20}, \\ -L_{\frac{p-3}{2}} & \text{if } p \equiv 7, 19 \pmod{20}, \\ -L_{\frac{p+1}{2}} & \text{if } p \equiv 9, 13 \pmod{20}, \\ L_{\frac{p+1}{2}} & \text{if } p \equiv 11 \pmod{20}, \\ L_{\frac{p-3}{2}} & \text{if } p \equiv 17 \pmod{20}. \end{cases}$$

Using Lemma 2.3 we have

$$\sum_{k=0}^{\lfloor \frac{p-1}{10} \rfloor} \frac{\binom{10k}{5k}}{(-4)^{5k}} \equiv \sum_{k=0}^{\lfloor \frac{p-1}{10} \rfloor} \binom{(p-1)/2}{5k} = \sum_{\substack{k=0 \\ 5|k}}^{(p-1)/2} \binom{(p-1)/2}{k} \pmod{p}.$$

Note that $L_{\frac{p-3}{2}} = L_{\frac{p+1}{2}} - L_{\frac{p-1}{2}}$. In [11, Corollaries 1 and 2], the author and Z.W. Sun determined $L_{\frac{p+1}{2}} \pmod{p}$. By the above and [11, Corollaries 1 and 2] we deduce the result.

Theorem 4.5. *Let $p \equiv 11 \pmod{20}$ be a prime. Then*

$$\sum_{k=0}^{\frac{p-11}{20}} \binom{20k}{10k} \frac{1}{4^{10k}} \equiv (-1)^{\frac{p+1}{4}} \frac{1}{10} \pmod{p}.$$

Proof. By Lemma 2.3 we have

$$10 \sum_{k=0}^{\frac{p-11}{20}} \binom{20k}{10k} \frac{1}{4^{10k}} - (-1)^{\frac{p+1}{4}} \equiv 10 \sum_{k=0}^{\frac{p-11}{20}} \binom{\frac{p-1}{2}}{10k} - 2^{\frac{p-1}{2}} = 10T_{0(10)}^{\frac{p-1}{2}} - 2^{\frac{p-1}{2}} \pmod{p}.$$

According to [11, Theorem 1 and Corollary 1],

$$10T_{0(10)}^{\frac{p-1}{2}} - 2^{\frac{p-1}{2}} = -2L_{\frac{p-1}{2}} \equiv 0 \pmod{p}.$$

Thus the result follows.

Theorem 4.6. *Let $p \equiv 13 \pmod{24}$ be a prime. Then*

$$\sum_{k=0}^{(p-13)/24} \binom{24k}{12k} \frac{1}{4^{12k}} \equiv \frac{1}{12} (1 - 2 \cdot 3^{\frac{p-1}{4}}) \pmod{p}$$

and

$$\sum_{k=0}^{(p-1)/12} \binom{12k}{6k} \frac{(-1)^k}{4^{6k}} \equiv 0 \pmod{p}.$$

Proof. Using Lemma 2.3 and the fact $2^{\frac{p-1}{2}} \equiv -1 \pmod{p}$ we have

$$12T_{0(12)}^{\frac{p-1}{2}} - 2^{\frac{p-1}{2}} = 12 \sum_{k=0}^{\lfloor p/24 \rfloor} \binom{\frac{p-1}{2}}{12k} - 2^{\frac{p-1}{2}} \equiv 12 \sum_{k=0}^{\lfloor p/24 \rfloor} \binom{24k}{12k} \frac{1}{(-4)^{12k}} + 1 \pmod{p}.$$

Since $\frac{p-3}{2} \equiv 5 \pmod{12}$, by (4.1) and [12, Theorem 2] we have

$$12T_{0(12)}^{\frac{p-3}{2}} - 2^{\frac{p-3}{2}} = 1 - 3^{\frac{p-1}{4}} + (-1)^{\frac{p-5}{8}} (2^{\frac{p-1}{4}} - V_{\frac{p-5}{4}}(4, 1))$$

and

$$12T_{-1(12)}^{\frac{p-3}{2}} - 2^{\frac{p-3}{2}} = 12T_{\frac{p-1}{2}(12)}^{\frac{p-3}{2}} - 2^{\frac{p-3}{2}} = 1 - 3^{\frac{p-1}{4}} - (-1)^{\frac{p-5}{8}}(2^{\frac{p-1}{4}} - V_{\frac{p-5}{4}}(4, 1)).$$

Thus, using (4.1) we obtain

$$12T_{0(12)}^{\frac{p-1}{2}} - 2^{\frac{p-1}{2}} = 12T_{0(12)}^{\frac{p-3}{2}} - 2^{\frac{p-3}{2}} + 12T_{-1(12)}^{\frac{p-3}{2}} - 2^{\frac{p-3}{2}} = 2(1 - 3^{\frac{p-1}{4}}).$$

Hence

$$\sum_{k=0}^{[p/24]} \binom{24k}{12k} \frac{1}{4^{12k}} \equiv \frac{1}{12}(-1 + 2(1 - 3^{\frac{p-1}{4}})) \pmod{p}.$$

Since

$$6 \sum_{k=0}^{(p-1)/12} \binom{12k}{6k} \frac{1}{4^{6k}} + 6 \sum_{k=0}^{(p-1)/12} \binom{12k}{6k} \frac{(-1)^k}{4^{6k}} = 12 \sum_{k=0}^{(p-13)/24} \binom{24k}{12k} \frac{1}{4^{12k}},$$

by the above and Theorem 4.3 we deduce the remaining result.

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